1. Show that there does not exist a function $f: \mathbb{R} \to \mathbb{R}$ continuous on $\mathbb{Q}$ but discontinuous on $\mathbb{R} \setminus \mathbb{Q}$

2. If $f: [0,1] \to \mathbb{R}$ is continuous and has only rational values, must f be a constant?

3. Let $f: [0,1] \to [0,1]$ be continuous. Show that f has a fixed point. ($c \in [0,1]$ is said to be fixed point of f if $f(c) = c$)

4. Let I be a closed bounded interval and let $f: I \to \mathbb{R}$ be a function with the property that $\forall x \in I$, function f is bounded on a neighborhood $V_\delta(x)$ of x. Prove that f is bounded on I.

5. Determine if the following functions are uniformly continuous:

(a) $f(x): (0,1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$

(b) $f(x): [0,\infty) \to \mathbb{R}$ " $f(x) = \sqrt{x}$

(c) $f(x): [0,M) \to \mathbb{R}$ " $f(x) = x^2$, $M > 0$

(d) $f(x): [0,\infty) \to \mathbb{R}$ " $f(x) = x^2$

(e) $f(x): \mathbb{R} \to \mathbb{R}$ " $f(x) = \frac{1}{x^2+1}$

(f) $f(x): \mathbb{R} \to \mathbb{R}$ " $f(x) = \cos(x^2)$

1. Assume there is such function f

Write $\mathbb{Q} = \{r_n\}_{n=1}^{\infty}$

Let $\varepsilon_i > 0$ and ε_i converge to 0. as i tends to ∞

Since f is cont. at r_1

$\exists$ nondegenerate closed interval I_1 around r_1 s.t.

$$|f(x) - f(r_1)| < \varepsilon_1 \quad \forall x \in I_1$$

Now define nested interval I_n as follow inductively:

if $r_{n+1} \notin I_n$, then $I_{n+1} = I_n$ (not always happen since $\mathbb{Q}$ is dense)

if $r_{n+1} \in I_n$, let I_{n+1} be interval around r_{n+1}

and $I_{n+1} \subset I_n$, and $r_n \notin I_{n+1}$ and s.t. $|f(x) - f(r_{n+1})| < \varepsilon_{n+1}$ if $x \in I_{n+1}$

Then each I_{n+1} satisfy, $r_n \notin I_{n+1}$.

Thus $\bigcap_n I_n$ contains no continuity point $\{r_n\}_{n=1}^{\infty}$

Since $\{I_n\}$ is nested intervals

$y \in \bigcap_n I_n$

$\vdash$ y is continuity point

Let $\varepsilon > 0$, choose n so that $I_n \neq I_{n-1}$ and $\varepsilon_n < \frac{\varepsilon}{2}$.

Then $y \in I_n$. If $z \in I_n$,

$$|f(y) - f(z)| \le |f(y) - f(r_n)| + |f(r_n) - f(z)| < \varepsilon_n + \varepsilon_n < \varepsilon$$

choose $V_\delta(y) \subset I_n$, then if $z \in V_\delta(y)$, $|f(y) - f(z)| < \varepsilon$

Then y is continuity point. Contradiction.

2. Suppose f is not constant.

Then $\exists\ y_1, y_2 \in \mathbb{Q}$ s.t. $f(x_1) = y_1$, $f(x_2) = y_2$

Then $\exists\ z \in \mathbb{R} \setminus \mathbb{Q}$, s.t. $y_1 < z < y_2$

By Intermediate Value Theorem,

$\exists\ t \in (x_1, x_2)$ s.t. $f(t) = z$

Contradiction.

Thus f is constant

3. If $f(0) = 0$ or $f(1) = 1$, done.

Then $0 < f(0)$ and $f(1) < 1$

Let $g(x) = x - f(x)$

g is continuous on $[0,1]$

and $g(0) < 0 < g(1)$

Then by IVT, $\exists\ t \in (0,1)$ s.t. $g(t) = 0$

Then $0 = t - f(t)$

$\Rightarrow f(t) = t$

4. Suppose f is not bounded on I

for any $n \in \mathbb{N}$, $\exists\, x_n \in I$ s.t. $|f(x_n)| > n$

since I is closed bounded,

x_n is bounded

By Bolzano-Weierstrass, $\exists$ convergent subsequence (x_{n_k}) of (x_n)

then $x_{n_k} \to x \in I$ since I is closed

$x \in I$, then $\exists\, V_\delta(x)$ s.t. f is bounded on $V_\delta(x)$

since $x_{n_k} \to x$, for above δ,

$\exists\, N \in \mathbb{N}$ s.t. if $k \geq N$, $|x_{n_k} - x| < \delta$

$\Rightarrow x_{n_k} \in V_\delta(x)$

Then $|f(x_{n_k})| < M$ for some M

But $M > |f(x_{n_k})| > n_k > M$, choose large n_k

contradiction

f is bounded on I.

5 a) No

b) Yes

c) Yes

d) No

e) Yes

f) no

Definition: Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, f is uniformly continuous on A if $\forall \varepsilon > 0$, $\exists\ \delta(\varepsilon) > 0$ s.t. if $x, u \in A$ with $|x-u| < \delta$, then $|f(x) - f(u)| < \varepsilon$.

5 a) Use Nonuniform Continuity Criteria

$\exists\ \varepsilon_0 > 0$ and $(x_n), (u_n)$ in A s.t. $\lim (x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0$ $\forall n \in \mathbb{N}$

if $x_n := \frac{1}{n}$, $u_n = \frac{1}{n+1}$, then $\lim_n (x_n - u_n) = 0$, but $|f(x_n) - f(u_n)| = 1$ $\forall n \in \mathbb{N}$

b) First consider $I = [0, 2]$, f is continuous on I and I is closed and bounded interval, then f is uniformly continuous on I

Consider $J = [1, \infty)$, if $x, u \in J$,

$$|f(x) - f(u)| = |\sqrt{x} - \sqrt{u}| = \frac{|x-u|}{\sqrt{x} + \sqrt{u}} \leq \frac{1}{2}|x-u|$$

f is Lipschitz function on J, then f is uniformly cont. on J

Since $[0, \infty) = I \cup J$, take $\delta = \min\{1, \delta_I, \delta_J\}$

f is uniformly continuous on $[0, \infty)$

5c) If $x, u \in [0, M)$

$$|f(x) - f(u)| = |x+u|\,|x-u| \leq 2M\,|x-u|$$

f is Lipschitz on $[0, M)$

Then f is uniformly continuous on $[0, M)$

d) if $x_n := n + \frac{1}{n}$, $u_n := n$

Then $\lim (x_n - u_n) = \lim \frac{1}{n} = 0$

but $|f(x_n) - f(u_n)| = |n^2 + \frac{1}{n^2} + 2 - n^2|$

$$= |2 + \frac{1}{n^2}|$$

$$> 2 \qquad \forall n \in \mathbb{N}$$

e) if $x, u \in \mathbb{R}$

$$|f(x) - f(u)| = \left|\frac{1}{1+x^2} - \frac{1}{1+u^2}\right|$$

$$= \left|\frac{1+u^2 - 1 - x^2}{(1+x^2)(1+u^2)}\right|$$

$$= |u-x| \frac{|u+x|}{|(1+x^2)(1+u^2)|}$$

$$\leq |u-x| \left(\frac{|u|}{(1+x^2)(1+u^2)} + \frac{|x|}{(1+u^2)(1+x^2)}\right)$$

$$\leq |u-x| \left(\frac{|u|}{1+u^2} + \frac{|x|}{1+x^2}\right)$$

$$\leq |u-x| \qquad \left(\text{since } (|x|-1)^2 \geq 0 \Rightarrow x^2+1 \geq 2|x| \; ; \right)$$

f is Lipschitz $\Rightarrow$ f is uniformly continuous.

f) if $x_n := \sqrt{n\pi}$, $u_n := \sqrt{n\pi + \frac{\pi}{2}}$, $\lim (x_n - u_n) = \lim_n \frac{-\frac{\pi}{2}}{\sqrt{n\pi} + \sqrt{n\pi + \frac{\pi}{2}}} = 0$

But $|f(x_n) - f(u_n)| = 1$